International Journal of Theoretical Physics, Vol. 16, No. 1 (1977), pp. 13-33

Segmented Structures

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Received: 16 *September* 1975

Abstract

A new mathematical structure is introduced, to be called *segmented structure*, defined by the assumption that for any pair of its points x, y a subset is defined, reasonably identifiable as "the segment joining x and y." The reason for such a research is the existence of a physically reasonable way of defining the segment joining two points of space-time. Thus, a powerful enough structure of space-time arises, which appears as deducible on physically reasonable grounds.

1. *Introduction*

In this work we want to describe a mathematical structure that turns out to be relevant in a possible approach to the problem of physical foundation of the space-time structure. Actually, we have endeavored to fully exploit the hypothesis that, given two events x and y in the "space-time," there is a physically reasonable way of assigning a set of events that can be thought of as a "segment" joining x and y. We have carefully avoided any "linearity" argument and have used only a few axioms about prolongability of our segments, to see what can be deduced therefrom; in fact we have succeeded in defining such concepts as dimension and manifold and in proving a number of reasonable results about them. We believe that such work has a remarkable geometric meaning too, as it exhibits what may be thought of as the purely geometric origin of many concepts, usual f.i. in Euclidean geometry. In particular, in a segmented structure it is possible to work out a reasonable theory of dimension, simplexes, etc.

1 Work supported in part by C.N.R.
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2. *Abstract Segmented Structures*

An abstract segmented structure is a pair (M, ρ) of a nonempty set M and a map $\rho: M \times M \to \mathscr{P}(M)$ satisfying axioms 1-8 that we will introduce at various stages throughout sections 2–4; for any pair $(x, y) \in M \times M$, the subset $\rho(x, y)$ is simply denoted by xy and called the *segment* joining x and y or simply segment xy.

Axiom 1. $(\forall x, y, x', y' \in M)\ [(xy = x'y') \Leftrightarrow (\{x, y\} = \{x', y'\})]$
Axiom 2. $(xy = \phi) \Leftrightarrow (x = y)$
Axiom 3.[3] $z \in xy \Leftrightarrow [xz \subset xy \text{ and } xz \neq \phi]$
Axiom 4. $[z \in xy \text{ and } u \in zy] \Rightarrow z \in xu$

We stop to draw a few consequences that will clarify the further axioms we eventually introduce. In order to avoid cumbersome nomenclature, we will talk of segmented structure also before introducing all axioms with the obvious meaning of "the structure satisfying all axioms introduced so far."

Lemma 2.1. If $z \in xy$, then $zy \subset xy$.

Proof: $z \in xy \Rightarrow z \in yx \Rightarrow yz \subset yx \Rightarrow zy \subset xy.\square$

Lemma 2.2. If $z \in xy$ and $v \in xz$, then $z \in vy$.

Proof: $(z \in xy \text{ and } v \in xz) \Rightarrow (z \in yx \text{ and } v \in zx) \Rightarrow z \in yv \Rightarrow z \in vy.\square$

Lemmas 2.1 and 2.2 are but symmetrizations of statements contained in axioms 3 and 4.

Theorem 2.1. If $x \neq y$, the relation "$z_1 \underset{xy}{\leqslant} z_2$ if $xz_1 \subseteq xz_2$ is a partial ordering in xy.

Proof: (i) $\underset{xy}{\leqslant}$ is obviously reflexive; (ii) $\underset{xy}{\leqslant}$ is transitive: $(z_1 \underset{xy}{\leqslant} z_2$ and $z_2 \underset{xy}{\leqslant} z_3) \Rightarrow (xz_1 \subseteq xz_2 \subseteq xz_3) \Rightarrow z_1 \underset{xy}{\leqslant} z_3$ for any $(z_1, z_2, z_3) \in xy \times xy \times xy$; (iii) $\underset{xy}{\leqslant}$ is antisymmetric: $(z_1 \underset{xy}{\leqslant} z_2$ and $z_2 \underset{xy}{\leqslant} z_1) \Rightarrow (xz_1 \subseteq xz_2$ and $xz_2 \subseteq xz_1) \Rightarrow xz_1 = xz_2$ which, by virtue of axiom 1, implies $z_1 = z_2.\square$

The structure of a segment is further analyzed by the following lemma:

Lemma 2.3. For any $z \in xy, xz \cap zy = \phi$.

Proof: Suppose $u \in xz \cap xy$; then $[z \in xy \text{ and } u \in xz] \Rightarrow xu \subset xz \Rightarrow u \underset{xy}{\leqslant} z$ and $u \neq z$; but $(z \in xy \text{ and } u \in zy) \Rightarrow z \in xu \Rightarrow xz \subset xu \Rightarrow z \underset{xy}{\leqslant} u$, which is clearly incompatible with the former.$\square$

Lemma 2.3 can be conveniently stated saying that, for any $z \in xy$, $xz \cup \{z\} \cup zy$ is a disjoint union, obviously contained in xy. So far, any nontrivial xy has the structure of a poset without minimum and without maximum. For, if z is any element belonging to xy, we can always find a $u \in xz$ (which

[3] Here and in the following $\subset$ means: "is properly contained in" and $\subseteq$ means "is contained in."

is nonempty) that obviously satisfies $u \underset{xy}{\leqslant} z$ and $u \neq z$. The symmetry of segments forbids analogously the existence of a maximum. It is easy to see that we can "extend" any nontrivial segment xy to a poset with minimum and maximum by adjoining the elements x and y (respectively, if the order is $\underset{xy}{\leqslant}$); such a subset is called *extended segment* $[xy]$; further, we set $[xx] = \{x\}$. We also have the following:

Lemma 2.4. If $x \neq y$ the segment xy contains at least a denumerable infinity of distinct points of M.

Proof: The proof is quite straightforward, as one easily defines a sequence of distinct points all belonging to xy, for instance by setting $z_0 \in xy$, $z_{i+1} \in xz_i$, for any integer i.□

So far, the structure of a segment is still allowed to be "not flat," in the sense that the order we have defined on a segment is not necessarily total. The following theorem shows how this requirement is equivalent to a strengthening of the statement of Lemma 2.3:

Theorem 2.2. If $x \neq y$, then $\underset{xy}{\leqslant}$ is a total ordering for xy iff for any $z \in xy(xz \cup \{z\} \cup zy = xy)$.

Proof: (a) Suppose $\forall z \in xy(xz \cup \{z\} \cup zy = xy)$; then, if z_1, z_2 are distinct points of xy, we either have $z_2 \in xz_1$ or $z_2 \in z_1y$; in the first case we have $z_2 \underset{xy}{\leqslant} z_1$, in the second $z_1 \in xz_2$, whence $z_1 \underset{xy}{\leqslant} z_2$ follows. Conversely, if the order is total we have, for any z and u belonging to xy: $(u = z$ or $u \underset{xy}{\leqslant} z$ or $z \underset{xy}{\leqslant} u) \Rightarrow (u = z$ or $xu \subset xz$ or $xz \subset xu) \Rightarrow (u = z$ or $u \in xz$ or $z \in xu) \Rightarrow$ $(u = z$ or $u \in xz$ or $u \in zy) \Rightarrow (xz \,\dot{\cup}\, \{z\} \cup zy = xy)$.□

Corollary: if $\underset{xy}{\leqslant}$ is total, then $\underset{yx}{\leqslant}$ is the dual ordering of $\underset{xy}{\leqslant}$: for, let z_1, z_2 be distinct points of xy and suppose $z_1 \underset{xy}{\leqslant} z_2$; then $xz_1 \subseteq xz_2$, which implies, by Theorem 2.2, $z_2y \subseteq z_1y$, i.e., $z_2 \underset{yx}{\leqslant} z_1$.□

Then we set as our fifth axiom the following:

Axiom 5. $(\forall z \in xy)\,(xz \cup \{z\} \cup zy \supseteq xy)$

Thus, any nontrivial segment xy in M is now a chain, for both $\underset{xy}{\leqslant}$ and $\underset{yx}{\leqslant}$; it contains a subchain isomorphic to Q, the chain of rational numbers with the natural ordering, since it does contain a countably infinite subset, dense in itself with respect to the induced ordering (Birkhoff, 1973). Such a subset is easily obtained by a procedure slightly more careful than the one exhibited in suggesting the proof of Lemma 2.4. We just need to consider, for a given nonempty segment z_0z_1, a point $z_{1/2} \in z_0z_1$, then $z_{1/4} \in xz_{1/2}$ and $z_{3/4} \in z_{1/2}y$ and so on, inserting between $z_{k/2^r}$ and $z_{k+1/2^r}$ a new point to be called $z_{2k+1/2^{r+1}}$. So, a subset is inductively defined, which is dense in itself and countable, so that the quoted theorem applies.

3. *Topological Arguments*

In this section we need to introduce a number of topological arguments. First, it is obvious that on each nontrivial segment the order topology is defined: a basis for it is the collection of all the subsegments z_1z_2 with $z_1, z_2 \in [xy]$. The extended segment $[xy]$ can be given a topology simply by taking as basis the said collection of open intervals of xy *and* the subsets of the type $\{x\} \cup xz_1 (z_1 \in xy), z_2y \cup \{y\}(z_2 \in xy)$; in this topology the closure of xy is evidently $[xy]$.

But, more important, we can define a topology (or rather various topologies) on the whole M. Let us give a previous natural definition:

> *Definition.* A subset $K \subseteq M$ is called *convex* if it contains xy whenever it contains x and y.

Denote by $\mathscr{K}$ the collection of all the convex subsets of M. Obviously any segment (either extended or not) is a convex set and the intersection of any family of convex subsets of M is a convex subset of M, so that it makes sense to define the *convex hull* CoA of any subset A of M, as

$$\mathrm{Co}A = \bigcap_{\substack{K \supset A \\ K \in \mathscr{K}}} K$$

The convex hull of a set x, y consisting of two distinct points coincides with the extended segment joining them.

Now, there are various topologies on M that appear as naturally definable. The first one, which we denote by $\mathscr{T}_1$, is defined by the following basis of subsets of M: $\mathscr{B}_1 = \{B \subset M \mid \forall x \in B, \forall z \in M, \exists y \in B \text{ s.t. } y \in xz, xy \subset B\}$ this means that a subset $B \in \mathscr{B}_1$ must contain, for each of its points, a segment starting from it and pointing to every "direction." It is easy to prove that $\mathscr{B}_1$ is indeed a basis for a topology; for, $M \in \mathscr{B}_1$ so that each $x \in M$ belongs to at least one element of $\mathscr{B}_1$; secondly, let B_1, B_2 belong to $\mathscr{B}_1$ and consider $B_3 = B_1 \cap B_2$; let x belong to B_3 and z to M: then there is a $y_1 \in B_1 \cap xz$ and $y_2 \in B_2 \cap xz$ such that $xy_1 \subset B_1$ and $xy_2 \subset B_2$; as the order is total on xz, we can suppose, say, $y_1 \underset{xz}{\leqslant} y_2$, then $xy_1 \subset B_1 \cap B_2 = B_3$.

It may be worth remarking that this topology is not so obvious, in the sense that, for instance in the standard case of the Euclidean space and Euclidean open segments in it, $\mathscr{T}_1$ does not coincide with the usual topology: It is strictly finer than that, as one easily realizes by considering, say in R^2, such subsets as the one sketched in Figure 1, which is indeed an open set for $\mathscr{T}_1$ and not an open set for the usual topology.

One also realizes that the crucial shortcoming with this topology is that it is not necessarily locally convex, as the exhibited example again shows. So it seems natural to introduce the following topology, which we shall call $\mathscr{T}_K$: consider the collection of subsets of M defined by $\mathscr{B}_K = \mathscr{B}_1 \cap \mathscr{K}$; $\mathscr{B}_K$ is a basis for a topology essentially because M is convex and the intersection of convex subsets is a convex subset; thus $\mathscr{B}_K$ defines a topology $\mathscr{T}_K$. $\mathscr{T}_K$ is locally convex and is in general coarser than $\mathscr{T}_1$.

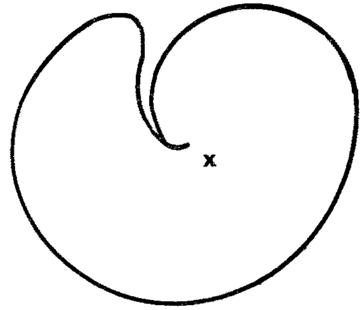

Figure 1.

A further topology that one might be tempted to introduce in M is the inductive topology $\mathscr{T}_I$ with respect to the family of all segments and of their natural injections in M: This is the finest topology for which all such injections are continuous; we will show that $\mathscr{T}_I$ is finer than $\mathscr{T}_1$, and in general strictly so. In order to assure the equivalence between these two topologies one further axiom is going to be needed, which amounts to requiring that every segment of M can be prolonged. We now prove the following theorem:

Theorem 3.1. For a segmented structure (satisfying axioms 1-5) we have $\mathscr{T}_I \supseteq \mathscr{T}_1$.

Proof: Let $A \subset M, A \in \mathscr{T}_1$; then, for any segment xy, $j_{xy}^{-1}(A) = xy \cap A$ is open in xy: for, if $xy \cap A \neq \phi$, take any $z \in xy \cap A$ and choose z' and z'' with the properties $zz' \subseteq zx \cap A$ and $zz'' \subseteq zy \cap A$, which is possible by virtue of the assumption $A \in \mathscr{T}_1$; then $z'z''$ is an open subset of xy containing z and we have $z'z'' = z'z \cup \{z\} \cup zz'' \subseteq (zx \cap A) \cup \{z\} \cup (zy \cap A) \subseteq xy \cap A$.□

In trying to prove the converse ($\mathscr{T}_I \subseteq \mathscr{T}_1$) one comes upon the difficulty that one cannot prolong a segment beyond either one of its extreme points; an example will clarify the matter: Consider the closed ball $\bar{B}_n$ of radius equal to 1 in R^n and define a segment xy in the usual (Euclidean) way; note that none of the segments with one end on the boundary S_{n-1} of $\bar{B}_n$ can be prolonged beyond that end; now consider a subset F of $\bar{B}_n$ defined as the union of a ball whose radius is strictly smaller than 1 with one arbitrary point z of S_{n-1}; this is not open for the topology $\mathscr{T}_1$, obviously; but *it is* open for $\mathscr{T}_I$, because any $j_{xy}^{-1}(F)$ is open in xy, as z cannot belong to it.

The last example could raise the suspicion that $\mathscr{T}_I$ being finer than $\mathscr{T}_1$ depends on the fact that there are points not belonging to any segment, rather than to the nonprolongability of some segments; that this is not so is shown by the following example: consider a closed (in the Euclidean topology) halfplane M in R^2 and let s be the straight line boundary of M; let the segments in M be defined in the obvious way: each point of M is contained in a segment (note, however, that for points on s the "direction" of such a segment is uniquely determined). A subset of M that is the union of an open subset (in the sense of the Euclidean topology) of the half-plane with an open segment lying on s is open for $\mathscr{T}_I$ and not for $\mathscr{T}_1$, as is easily seen; furthermore, with respect to $\mathscr{T}_I$, M is even a nonconnected topological space, in that it can be obtained as the union of two of its disjoint open subsets: s and its complement in M (that this is not so with respect to $\mathscr{T}_1$ is immediately seen).

On the other hand, if we postulate the following further axiom the situation actually improves:

Axiom 6. If xy is a nontrivial segment of M, then there exists a $z \in M$ such that $x \in zy$ (hence $xy \subset zy$).

In fact we now prove that under Axioms 1-6 the following result holds:

Theorem 3.2. The two topologies $\mathscr{T}_I$ and $\mathscr{T}_1$ on M are equivalent.

Proof: We have already proved that $\mathscr{T}_1 \subseteq \mathscr{T}_I$. As for the converse, suppose $A \in \mathscr{T}_I$; that amounts to saying that $xy \cap A$ is open in xy for any segment $xy \subseteq M$. Let $x \in A$ and $z \in M$; consider the segment xz and, by virtue of Axiom 6, choose a point $y \in M$ such that $x \in yz; yz \cap A$ is open in yz and contains x; it will then contain a segment $z'z''$ such that $x \in z'z'' \subset yz \cap A$; it follows that, say, $xz'' \subseteq xz \cap A$, so that $A \in \mathscr{T}_1$.□

As for general properties of the topologies so far introduced it is easy to see that $\mathscr{T}_1$ (and hence $\mathscr{T}_I$) is a T_1 topology: Each subset of M of the form $\{x\} (x \in M)$ is closed; that $M - \{x\}$ is open for $\mathscr{T}_1$ is immediately seen, since for any $z \in M - \{x\}$, the nonempty segment zx contains a subsegment zy (take any $y \in zx$) which is contained in $M - \{x\}$. As for $\mathscr{T}_K$, we do not have at this point any statement, but the question will be examined more closely in Section 5.

4. *Geometrical Axioms*

In order to develop a fairly reasonable geometry of polyhedra in a segmented structure, we have to introduce some further axioms, the first of which amounts to requiring the uniqueness of the prolongation of a given segment. It can be precisely formulated in two points, as follows:

Axiom 7. If x, y, z, w, are four distinct points of M, then (a) $y \in xz \cap xw$ implies either $z \in xw$ or $w \in xz$; and (b) $x \in yz$ and $z \in xw$ imply $x \in yw$.

While (a) means that two prolongations of the same segment must be one contained in the other, (b) means that the prolongation of a subsegment of a given segment is a prolongation of the whole segment as well.

Axioms 7(a) and 7(b) will be used throughout as they will prove crucial in many proofs; it is not easy to see what happens if we relax even only one of them.

Next, we make the following definition of *join* and extended join.

Definition. If A and B are nonempty subsets of M, the *join* (respectively, *extended join*) of A and B, denoted by AB (respectively, $[AB]$), is the subset of M consisting of all segments (extended segments) joining a point of A with a point of B:

$$AB := \bigcup_{\substack{x\in A\\ y\in B}} xy \text{ (respectively, } [AB] := \bigcup_{\substack{x\in A\\ y\in B}} [xy])$$

With this definition we have, obviously,

$$A \cup B \subseteq [AB], \qquad AB \subseteq [AB], \qquad AB = BA, \qquad A \subseteq B \Rightarrow AC \subseteq BC,$$

$$AA \subseteq A \qquad \text{iff } A \text{ is convex}$$

$\{x\}\ \{y\} = xy$, $[\{x\}\ \{y\}] = [xy]$ for any couple of points x, y of M; besides, $[AB]$ is never empty, and AB is empty if and only if $A = B = \{x\}$ with $x \in M$. In terms of join, a useful characterization of openness is the following:

Theorem 4.1. If a set $A \subseteq M$ is open (with respect to $\mathscr{T}_1 \equiv \mathscr{T}_I$) then $A \subseteq A\{x\}, \forall x \in M$.

Proof: Suppose A open in M; then if $y \in A$ and $z \in M$, find a z' such that $y \in z'z$; since A is open, there exist $z'' \in A \cap z'y$; then $y \in z''z \subseteq A\{z\}$.□ The converse is not true, as the example of $\mathbb{Q} \subset \mathbb{R}$ shows (with obvious definitions of segments).

An interesting and natural property of the join is the one assumed with the following axiom:

Axiom 8. If x, y, z are distinct points of M,

$$(\{x\}\{y\})\{z\} = \{x\}(\{y\}\{z\})$$

We will often refer to this property as *associativity.*

Before showing what happens with extended joins, note that associativity allows for a sensible definition of *triangle* in M:

Definition. Call triangle with vertices x, y, z, (*distinct* points of M) the subset xyz of M given by $(\{x\}\{y\})\ \{z\}$. Analogously call extended triangle $[xyz]$ the subset $[[\{x\}\{y\}]\ \{z\}]$.[4]

It is immediately seen that, by virtue of Axiom 8, the definition of xyz is completely symmetric with respect to the three points x, y, z. As for extended things, the following theorem obtains:

Theorem 4.2. If x, y, z are distinct points of M, then

$$[[\{x\}\{y\}]\ \{z\}] = [\{x\}[\{y\}\{z\}]]$$

Proof: The two subsets of M that we must show to be equal contain the common nonempty subset $\{x\}(\{y\}\{z\})$; an easy direct inspection shows that points of $[[\{x\}\{y\}]\ \{z\}]$ that are not in $(\{x\}\{y\})\{z\}$ may only lie on $[xy]$ or on $[xz]$ or on $[yz]$ and these sets are contained in $[\{x\}[\{y\}\{z\}]]$; thus the proof is completed.□

4 From now on, by $x, y, z, \ldots$ we will always mean points of M, unless the contrary is explicitly stated.

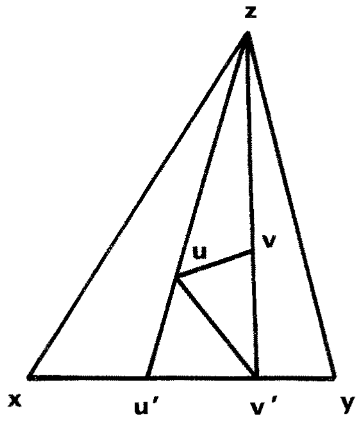

Figure 2.

Theorem 4.3. Every triangle is convex.

Proof: Let xyz be a triangle of M and u and v two distinct points of xyz; this means that u' and v' exist, belonging to xy (say) such that $u \in zu'$ and $v \in zv'$ (see Figure 2). In order to show that a point w of uv also belongs to xyz consider the triangle $zu'v' \subseteq zxy$. Remark that, by associativity $zuv' = uzv' \subseteq u'zv' = zu'v'$, as $w \in zuv'$, $w \in zxy$ follows.□

It may be interesting to show an explicit example of a structure satisfying Axioms 1-7 but not Axiom 8, thus showing the independence of the latter from the former. Let $M' = \mathbb{R}^2$ be given a segmented structure by the following prescription: The segment joining (x_1, y_0) with (x_2, y_0), is $\{(x, y) \in \mathbb{R}^2$ s.t. $x_1 < x < x_2, y = y_0\}$; the segment joining (x_1, y_1) with (x_2, y_2), say $y_1 > y_2$, is the set $\{(x, y) \in \mathbb{R}^2$ s.t. $x_1 < x, y = y_1\} \cup \{(x, y) \in \mathbb{R}^2$ s.t. $y_1 > y > y_2\} \cup \{(x, y) \in \mathbb{R}^2$ s.t. $x < x_2, y = y_2\}$; in other words segments are open chains in $\mathbb{R}^2$ ordered by the lexicographic order. So far, M' is a segmented structure satisfying Axioms 1-7, and also Axiom 8, trivially as any triple of points of M consists of collinear points, so that every triangle is trivial. Consider now the set $M = M' \times R \equiv R^3$ with the following definition of segments: If two distinct points have the same "third coordinate," say (x, y, z), (x', y', z), define the segment joining them via the definition previously given for M' and the obvious bijection of M' onto $M' \times \{z\}$; if two points do *not* have the same third coordinate, define the segment joining them as the usual Euclidean open segment; one easily checks that this structure satisfies Axioms 1-7. On the other hand $((x, y, z)(x', y', z))\ \{x'', y'', z'\}$ contains properly, whenever $z \neq z'$, $\{(x, y, z)\}((x', y', z)(x'', y'', z'))$ as one easily sees.

Further consequences of the associativity are the following:

Lemma 4.1. $A(BC) = (AB)C$ for any triple A, B, C of subsets of M.

Proof.

$$A(BC) = \bigcup_{x \in A} (\{x\}(\bigcup_{\substack{y \in B \\ z \in C}} \{y\}\{z\})) = \bigcup_{\substack{x \in A \\ y \in B \\ z \in C}} (\{x\}(\{y\}\{z\}))$$

$$= \bigcup_{\substack{x \in A \\ y \in B \\ z \in C}} ((\{x\}\{y\})\{z\}) = (AB)C$$

Lemma 4.2. $\{x\}A$ is convex whenever A is convex.

The proof is elementary.

Lemma 4.3. $(xy)K$ is convex whenever K is convex.

Proof. $(xy)K = (\{x\}\{y\})K = \{x\}(\{y\}K)$, which is convex, as Lemma 4.2 shows.

Lemma 4.4. K_1K_2 is convex whenever K_1 and K_2 are convex.

Proof. Let $\{x, y\} \subseteq K_1K_2$; this means that there exist x_1, y_1 in K_1 and x_2, y_2 in K_2 such that $x \in x_1x_2, y \in y_1y_2$; then $x \in \{x_1\}K_2, y \in \{y_1\}K_2$, and this implies, by the very definition of join, $xy \subseteq (\{x_1\}K_2)(\{y_1\}K_2) \subseteq \{x_1\}\{y_1\}K_2$, where the relation $K_2K_2 \subseteq K_2$ has been used. As $\{x_1\}\{y_1\}K_2 \subseteq K_1K_2$ (as K_1 is convex) the proof is completed.□

Completely analogous lemmas hold in which joins are replaced by extended joins.

We now have to go into a number of results concerning triangles: The aim is to exhibit the geometrical properties of a segmented structure satisfying Axioms 1–8 and to prepare the background for the main theorems on manifolds. We will call xyz a nontrivial triangle if x, y, and z are not collinear, i.e., if none of them lies on the segment joining the other two. Note that, by virtue of Axiom 7(b), if two points z and z' of M lie on the prolongation of a nonempty segment xy, then they are collinear with both x and y.

Lemma 4.5. If xyz is a nontrivial triangle, and if $u \in yz$, then $xyz = xyu \cup xu \cup xuz$ and the union is disjoint.

Proof. The only nontrivial thing to prove is disjointedness. This is where nontriviality of the triangle comes in. For, suppose $w \in xyu \cap xu$ (see Figure 3); then, as $w \in xyu$, there must be a point $u' \in yu$ such that $w \in xu'$; but then $w \in xu \cap xu'$ implies either $u \in xu'$ or $u' \in xu$, and this leads easily to the collinearity of x, y, and z. Completely analogous is the proof for disjointedness of xu and xuz and of xyu and xuz.□

Remark. If the extended triangle is considered, an obvious argument leads to the conclusion that $[xyz] = [xyu] \cup [xuz]$ and that $[xyu] \cap [xuz] = [xu]$.

A very important result is now the following theorem:

Theorem 4.4. If $u \in xy$ and $v \in yz$, then $xv \cap zu \neq \phi$.

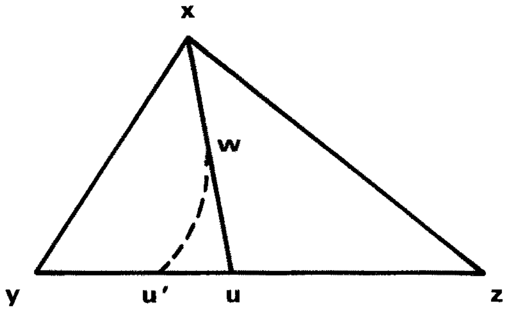

Figure 3.

Proof. As xv is nonempty (unless $x \in yz$, in which case the theorem is trivial) choose a point r on it: r belongs to xyz; if $r \in zu$ our task is completed; (see Figure 4) by Lemma 4.5 the only other possibilities are $r \in zxu$ or $r \in zuy$. Consider the first case: Then, by Axiom 8, there is a point $r' \in zu$ such that $r \in xr'$; by Axiom 7, as $r \in xr' \cap xv$, either $v \in xr'$ (which is false as $v \notin xyz$) or $r' \in xv$, so that $r' \in zu \cap xv$. If, on the other hand, $r \in zuy$ (then call it r'') consider the triangle zvu; obviously $r'' \in zvu$ and not to vu or to vuy, as is seen applying the remark following Lemma 4.5 to the triangle xvy. Then the above argument works again and shows that there is a point r' on $zu \cap vx$. □

If now Axiom 7(b) is taken into account, it is easy to see that for a nontrivial triangle the said intersection consists of exactly one point.

We proceed now with the following lemmas:

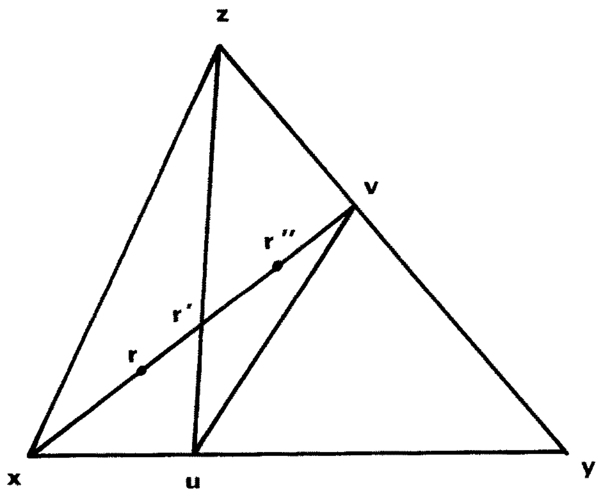

Figure 4.

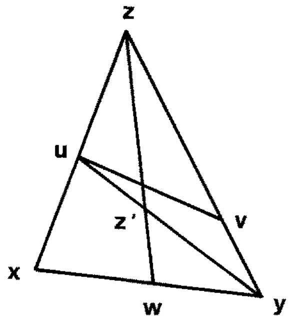

Figure 5.

Lemma 4.6. If $u \in xz$, $v \in yz$, $w \in xy$, then $wz \cap uv \neq \phi$.

Proof. (See Figure 5). By Theorem 4.4 there is a $z' \in zw \cap uy$, so that the triangle zuy can be considered and associativity applied to it: $zz' \cap uv$ is not empty and so is $wz \cap uv$.□ Again nontriviality of xyz would lead to $wz \cap uv$ consisting of just one point.

Lemma 4.7. If $y \in xp$ and $v \in xyz$, then $pv \cap yz \neq \phi$.

Proof. (see Figure 6). As $v \in xyz$, there exists $w \in xy$ such that $v \in zw$; applying Theorem 4.4 to the triangle pwz gives the result.□

Remark. If v' belongs to the side xz of xyz ("opposite" to p), the lemma still holds, again by virtue of Theorem 4.4 applied to the triangle pxz.

A basical lemma is the following:

Lemma 4.8. If u, v, w belong to xyz, and $v \in up$, then $wp \cap xyz \neq \phi$.

Proof. (See Figure 7). First it must be proved that there are two points u' and v' belonging to distinct sides (one of the points possibly coinciding with a vertex) of the triangle such that $uv \subset u'v'$; this can be easily done as follows:

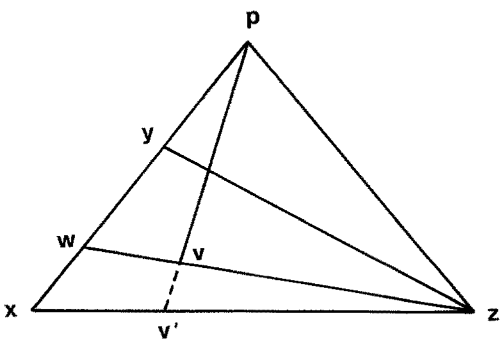

Figure 6.

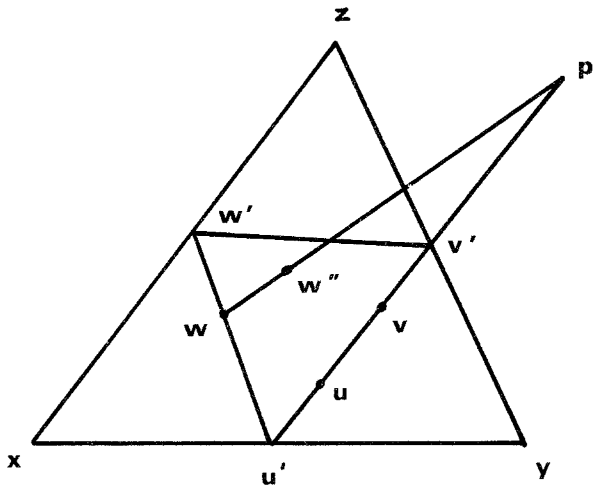

Figure 7.

Divide up xyz into three triangles by the segments ux, uy, and uz, then check where v lies: If not on one of the three segments named (in which case the result follows immediately), say, $v \in uzy$, apply associativity to find $v' \in zy$, then repeat the argument starting from v and conclude [here Axiom 7(b) is crucial] that $u' \in xy$ and $v' \in zy$ must exist such that $uv \subset u'v'$. If now $w \in zu'y$ (that is either to $zu'v'$ or to $u'v'$ or to $u'v'y$) the result follows by Lemma 4.7. If $w \in xu'z$ find $w' \in xz$ such that $w \in u'w'$. Now the result follows by applying the remark following Lemma 4.7 to the triangle $u'v'w'$. Thus, a point w'' exists belonging to $u'v'w'$, and then to xyz, such that $w'' \in wp$ and the lemma is proved. Note that, by Theorem 4.3, $ww'' \subseteq xyz$.□

In order to generalize these lemmas to tetrahedra and polyhedra we need some previous definitions and remarks.

From what we have seen with Theorems 4.2 and 4.3 it is evident that every triangle is convex, every extended triangle is convex, and the equality

$$[xyz] = \mathrm{Co}\{x, y, z\}$$

holds true for any triple (x, y, z) of points of M.

Now we define, by induction,

$$x_1 \cdots x_n = (x_1 \cdots x_{n-1})\{x_n\}$$

and, analogously,

$$[x_1 \cdots x_n] = [[x_1 \cdots x_{n-1}]\{x_n\}]$$

that is, what we can reasonably call the n-polyhedron and extended n-polyhedron of vertices $x_1, \ldots, x_n$; it is easy to generalize the above equality to the following:

$$[x_1 \cdots x_n] = \mathrm{Co}\{x_1, \ldots, x_n\}$$

indeed, that $[x_1 \cdots x_n] \subseteq \mathrm{Co}\{x_1, \ldots, x_n\}$ is obvious from the definition of the left-hand side, and that $[x_1 \cdots x_n] \supseteq \mathrm{Co}\{x_1, \ldots, x_n\}$ is shown by proving, by induction, that $[x_1 \cdots x_n]$ is convex. Suppose that $[x_1 \cdots x_n]$ is convex for any n-tuple $(x_1, \ldots, x_n)$; then Lemma 4.2 (generalized to extended joins) shows that $[x_1 \cdots x_{n+1}] = [[x_1 \cdots x_n]\{x_{n+1}\}]$ is convex too. From now on, then, the symbols $[x_1 \cdots x_n]$ and $\mathrm{Co}\{x_1, \ldots, x_n\}$ can be interchangeably used; notice, in particular, that, if $y \in \mathrm{Co}\{x_1, \ldots, x_n\}$, a point $y' \in \mathrm{Co}\{x_1, \ldots, x_{n-1}\}$ can be found such that $y \in [y'x_n]$. On the other hand, it is true by definition that if $y \in x_1 \cdots x_n$ there exists a point $y' \in x_1 \cdots x_{n-1}$ such that $y \in y'x_n$.

We now generalize Lemma 4.7 to the case of a 4-polyhedron, as this case proves useful for the general case too.

Lemma 4.9. Let x_1, x_2, x_3, x_4 be distinct points, $y \in x_1x_2x_3x_4$ and $x_2 \in x_1z$; then $yz \cap x_1x_2x_3x_4 \neq \phi$ and is a convex subset of yz.

Proof. (See Figure 8). As $y \in x_1x_2x_3x_4$, there exists $y' \in x_2x_3x_4$ such that $y \in x_1y'$; consider the triangle $x_1y'z$: Theorem 4.4 states that there exists $y'' \in x_2y' \cap zy$; then $y''y \subseteq x_1y''y' \subseteq x_1x_2x_3x_4$; $y''y$ can be assumed nonempty because $y = y''$ would imply collinearity of x_1, x_2, z, y, y', and the proof would be trivial.□

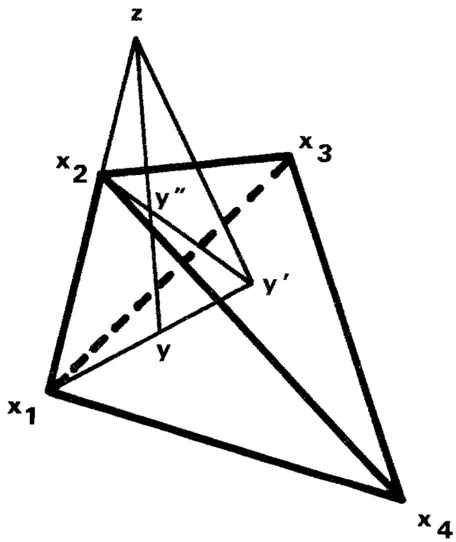

Figure 8.

Now, let K be a convex subset of M; let K^* be the following set: $K^* = \{x \in M, \text{ s.t. } \exists\, x_1, x_2 \in K \text{ s.t. } x_1 \in xx_2\}$; then the following lemma is important:

> *Lemma 4.10.* For any K, convex subset of M, K^* is convex and, if K contains at least two distinct points, $K \subseteq K^*$.

Proof. (See Figure 9). Let $x, y \in K^*$; then x_1, x_2, y_1, y_2, exist in K such that $x_2 \in x_1x, y_2 \in y_1y$; we have to show that $xy \subset K^*$: Let $z \in xy$ and let u be any point of $x_1x_2y_1y_2$. As K is convex we have also $[x_1x_2y_1y_2] = \mathrm{Co}\{x_1, x_2, y_1, y_2\} \subseteq K$; by Lemma 4.9 u' and u'' exist such that $u' \in xu, u'' \in yu$ and u', u'' belonging to $x_1x_2y_1y_2$; consider now the triangle uxy; by Lemma 4.6 $u'u'' \cap uz$ is nonempty so that a point v exists such that $v \in uz$ and $v \in u'u'' \subseteq x_1x_2y_1y_2 \subseteq K$. As for the second statement of the lemma, note that if $K = \{x\}$, then $K^* = \phi$; but that if K contains at least two points, from $x \in K$, $xy \subseteq K$ follows for some $y \neq x$, and then x, which lies on a (actually on each "left") prolongation of xy, belongs to K^*□

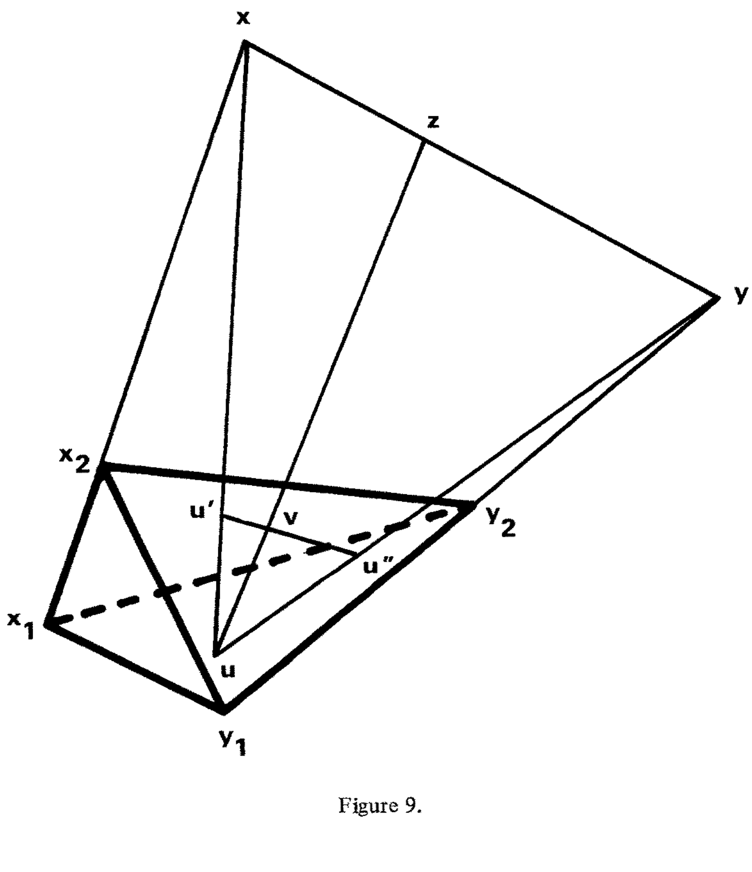

Figure 9.

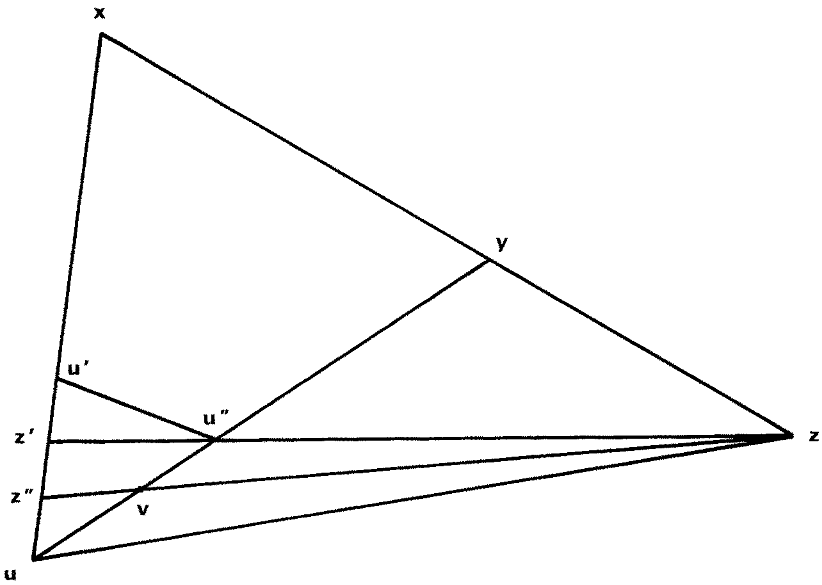

Figure 10.

The second fundamental property of the operation $K \to K^*$ is idempotency, proved by the following lemma:

Lemma 4.11. If $K \subseteq M$ is convex, then $K^{**} = (K^*)^* = K^*$.

Proof. (See Figure 10). Obviously we just need to show $K^{**} \subseteq K^*$. If z belongs to K^{**}, it means that x, y exist, belonging to K^*, such that, say, $y \in xz$; thus we reproduce the situation of Lemma 4.10 as for points x and y, adjoining now z. Remember that the whole $uu'u''$ is obviously contained in K; then consider uxz: A point z' exists on ux such that $u'' \in zz'$; take now a z'' belonging to the nonempty set $uu' \cap uz'$; we have $z'' \in uu' \subseteq K$; besides, as $z'' \in uz'$ a point $v \in uu'' \subset K$ exists such that $v \in z''z$, and this implies $z \in K^*$.□

5. *Manifold Theory in Segmented Structures*

We now want to work out a manifold theory in a segmented structure; it will turn out that many results familiar from the Euclidean case generalize to a segmented structure. First we need some natural definitions:

> *Definition.* A subset V of M is a *manifold* in M if $V = \operatorname{Co} V$ and if V contains the extended prolongations of all the segments lying in V; this may equivalently be formulated by saying that if x, y are distinct points in M and if xy contains two distinct points belonging to V, then $[xy] \subset V$.

Obvious consequences are that M itself is a manifold, every subset of M consisting of a single point $\{x\}$ is a manifold in M, and that if V_α is a manifold for each $\alpha \in A$, then $\bigcap_\alpha V_\alpha$ is a manifold.
This allows us to define the manifold *spanned* by a subset A of M, as the intersection of all manifolds containing A; we will denote it by $VA =: \bigcap_{V \supseteq A} V$, V is a manifold.

> *Definition.* n distinct points $(x_1, \ldots, x_n)$ of M are said to be *Co independent* (respectively, *V independent*) if the relation $\{x_1, \ldots, x_n\} \subset \mathrm{Co}\{y_1, \ldots, y_r\}$ (respectively $\{x_1, \ldots, x_n\} \subset V\{y_1, \ldots, y_r\}$ is not satisfied for any r-tuple $(y_1, \ldots, y_r)$ of points of M, with $r \leqslant n - 1$.

It is obvious that V independence implies Co independence, as $\mathrm{Co}\{y_1, \ldots, y_r\} \subseteq V\{y_1, \ldots, y_r\}$ and the inclusion is, in general, strict.

Accordingly, we introduce the following definition of dimension for a segmented structure M; the definition makes sense for any convex subset K of M [such that Axioms 1-8 are satisfied in K (this condition is nontrivial for Axiom 6)], in that K itself may be thought of as a segmented structure.

> *Definition.* We say that a segmented structure M has a finite Co dimension (V dimension) and that this is n, where n is a positive integer, if there is in M a $(n + 1)$-tuple of Co-independent (V-independent) points but no $(n + 2)$-tuple of such points.

We will denote by n_C the Co dimension and by n_V the V dimension. If for any n there is an $(n + 1)$-tuple of Co-independent (V-independent) points, we say that M has infinite Co dimension (V-dimension).

The relation between Co and V independence implies that if n_C and n_V both exist finite, then $n_C \geqslant n_V$ holds true; as for other possible cases, we can say that if the V dimension is infinite then so is the Co dimension. If, for a given segmented structure M, both the dimensions exist finite, we call the positive or null integer $n_C - n_V$ the *complexity* of M; if n_V is finite and the Co dimension is infinite we say that the complexity is infinite; the complexity is not defined if both the dimensions are infinite.

An elementary example will clarify the significance of such a definition: Consider a plane convex polygon with n sides ($n \geqslant 3$) and such that three of its vertices are never collinear; its interior M can be given the obvious Euclidean segmented structure; M has complexity equal (its Co dimension being equal to $n - 1$) to $n - 3$, as is easy to check; analogously B_2, the open unit ball in R^2, has infinite complexity; these trivial examples elucidate, by the way, the algebraic–geometric relevance of the notion of complexity.

We now proceed to establish the main properties of manifolds with the aim of getting a better description of the relations between topology in M and its (to be defined) simplexes. A few lemmas will be needed.

By independence and independent we will always mean, unless the contrary is explicitly stated, V independence and V independent, respectively.

> *Lemma 5.1.* If K is a convex subset of M containing at least two distinct points, we have $VK = K^*$.

Proof. (a) $K^* \subseteq VK$ because $z \in K^*$ implies that z lies on the prolongation of a segment of K and hence $z \in VK$. (b) $VK \subseteq K^*$ because K^* is a non-empty convex subset containing K, such that $K^* = K^{**}$, and hence is a manifold.□

Lemma 5.2. If A is any subset of M containing at least two distinct points, we have $VA = (\text{Co}\, A)^*$.

Proof. From Lemma 5.1 it follows $V(\text{Co}A) = (\text{Co}A)^*$. On the other hand, $V(\text{Co}A) \supseteq VA$ and also $V(\text{Co}A) \supseteq VA$ since $\text{Co}A \subseteq VA$ and $V(VA) = VA$.□ Take, in particular, $A = x_1 \cdots x_n$: We have $V\{x_1, \ldots, x_n\} = [x_1 \cdots x_n]^* = (\text{Co}\,\{x_1, \ldots, x_n\})^*$. According to our definitions, it is obvious that, whenever $(x_1, \ldots, x_n)$ is an n-tuple of independent points, $V\{x_1, \ldots, x_n\}$ has dimension $n - 1$; for, it contains the n-tuple $(x_1, \ldots, x_n)$ of independent points and it cannot contain an $(n + 1)$-tuple $(y_1, \ldots, y_{n+1})$ of independent points by the definition of independence.

Lemma 5.3. If $\{y_1, \ldots, y_n\}$ is an n-tuple of independent points, then we have $y_i \notin V\{y_1, \ldots, \hat{y}_i, \ldots, y_n\}$ for any i from 1 to n. (The caret over y_i means that y_i has to be omitted).

Proof. The proof is obvious: If $y_i \in V\{y_1, \ldots, \hat{y}_i, \ldots, y_n\}$ then $\{y_1, \ldots, y_n\} \subseteq V\{y_1, \ldots, \hat{y}_i, \ldots, y_n\}$, which cannot be, owing to the independence assumption.□

Lemma 5.4. If $(y_1, \ldots, y_{n-1})$ is an $(n - 1)$-tuple of independent points and y_n and z exist such that $y_n \notin V\{y_1, \ldots, y_{n-1}\}$ and $\{y_1, \ldots, y_n\} \subset V\{y_1, \ldots, y_{n-1}, z\}$, then $V\{y_1, \ldots, y_n\} = V\{y_1, \ldots, y_{n-1}, z\}$.

Proof. For the sake of brevity, put $W \equiv V\{y_1, \ldots, y_{n-1}\}$, $W_{y_n} \equiv V\{y_1, \ldots, y_n\}$, $W_z \equiv V\{y_1, \ldots, y_{n-1}, z\}$. We already know that $W_z = (\{z\}W)^*$. Then, as $y_n \in W_z$, we can find p and q in $\{z\}W$ such that (unless $y_n \in [pq]$, which is the most favorable case and leads to an immediate proof), say, $p \in y_n q$ (see Figure 11). If now p' and q' are those points of W such that $p \in [zp']$ and $q \in [zq']$, consider the triangle $zp'q'$: $p'q$ and $q'p$ meet in r; on the other side r belongs to the triangle $y_n qq'$, so that $y_n q'$ and qp' meet in a point r', which cannot coincide with p' because $y_n \notin W$ and because therefore q can always be chosen off W as well. Then q, which is on a prolongation of $p'r'$ (lying in W_{y_n}), and z, which is on a prolongation of qq' (lying in W_{y_n}) are in W_{y_n}. Therefore $W_z \subseteq W_{y_n}$ and hence the conclusion.□

Corollary. Owing to the Lemma 5.3, Lemma 5.4 is *a fortiori* true if the first two assumptions are replaced by the assumption "If $(y_1, \ldots, y_n)$ is an n-tuple of independent points of M."

Lemma 5.5. Suppose that for a given r, with $1 \leqslant r \leqslant n - 1$ the following is true: If $(y_1, \ldots, y_{n-1})$ is an $(n - 1)$-tuple of independent points, if $y_n \notin V\{y_1, \ldots, y_{n-1}\}$, and if $z_1, \ldots, z_r$ are points such that $\{y_1, \ldots, y_{n-1}, y_n\} \subseteq V\{y_1, \ldots, y_{n-r}, z_1, \ldots, z_r\}$, then $V\{y_1, \ldots, y_n\} =$

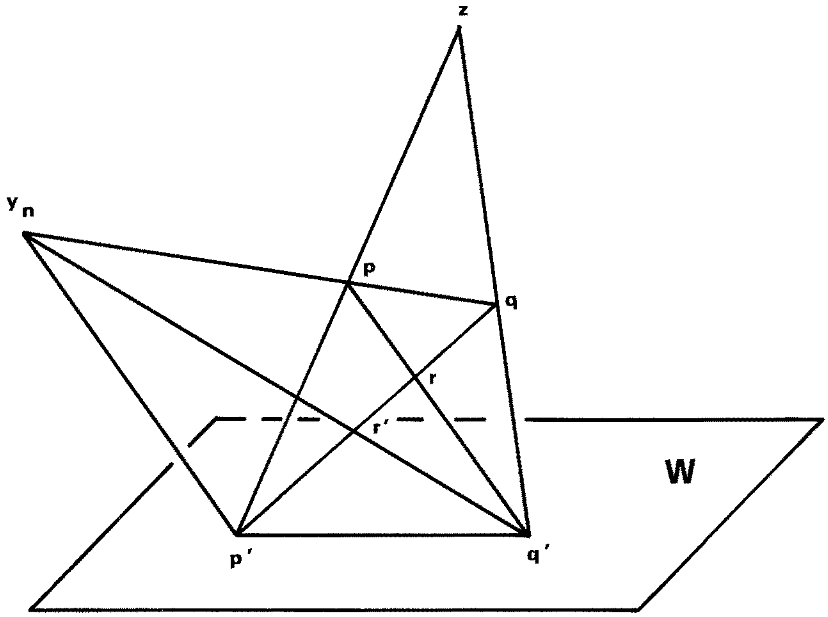

Figure 11.

$V\{y_1, \ldots, y_{n-r}, z_1, \ldots, z_r\}$. Then the same statement with r replaced by $r + 1$ is true.

Proof. Let $\{y_1, \ldots, y_{n-1}\}$ be $n - 1$ independent points, let $y_n \notin V\{y_1, \ldots, y_{n-1}\}$, and let $z_1, \ldots, z_{r+1}$ be such that $\{y_1, \ldots, y_n\} \subseteq V\{y_1, \ldots, y_{n-r-1}, z_1, \ldots, z_{r+1}\}$. Note that $\{z_1, \ldots, z_{r+1}\} \subseteq V\{y_1, \ldots, y_{n-1}\}$ is certainly false, because it would imply $y_n \in V\{y_1, \ldots, y_n\} \subseteq V\{y_1, \ldots, y_{n-r-1}, z_1, \ldots, z_{r+1}\} \subseteq V\{y_1, \ldots, y_{n-1}\}$, and this is contrary to the assumption. Then there is at least one of the z_i's, say z_1, not belonging to $V\{y_1, \ldots, y_{n-1}\}$. Note now that we have $\{y_1, \ldots, y_{n-1}, z_1\} \subseteq V\{y_1, \ldots, y_{n-r-1}, z_1, \ldots, z_{r+1}\}$; the set of points on the left-hand side and the set of points on the right-hand side differ by (at most) r points (that is, the second set contains $z_2, \ldots, z_{r+1}$ in place of $y_{n-r}, \ldots, y_{n-1}$), so that we are in a position to apply the result which has been assumed true. Then we conclude: $V\{y_1, \ldots, y_{n-1}, z_1\} = V\{y_1, \ldots, y_{n-r-1}, z_1, \ldots, z_{r+1}\}$; and, again taking into account the assumption $\{y_1, \ldots, y_n\} \subseteq V\{y_1, \ldots, y_{n-r-1}, z_1, \ldots, z_{r+1}\}$, we get $\{y_1, \ldots, y_n\} \subseteq V\{y_1, \ldots, y_{n-1}, z_1\}$; this time, as $y_n \notin V\{y_1, \ldots, y_{n-1}\}$, we can use Lemma 5.4 and get $V\{y_1, \ldots, y_n\} = V\{y_1, \ldots, y_{n-1}, z_1\} = V\{y_1, \ldots, y_{n-r-1}, z_1, \ldots, z_{r+1}\}$.□
The following theorem is now obvious:

Theorem 5.1. If $\{y_1, \ldots, y_{n-1}\}$ is an $(n - 1)$-tuple of independent points, if $y_n \notin V\{y_1, \ldots, y_{n-1}\}$ and $z_1, \ldots, z_n$ are such that $\{y_1, \ldots, y_n\} \subset V\{z_1, \ldots, z_n\}$, then $V\{y_1, \ldots, y_n\} = V\{z_1, \ldots, z_n\}$.

Corollary. Owing to Lemma 5.3, Theorem 5.1 is obviously true if the first two assumptions are replaced by the assumption "if $\{y_1, \ldots, y_n\}$ is an n-tuple of independent points."

Corollary; The converse of Lemma 5.3 is now provable; i.e., if $\{y_1, \ldots, y_{n-1}\}$ is an $(n-1)$-tuple of independent points, if y_n is such that $y_n \notin V\{y_1, \ldots, y_{n-1}\}$, then $y_1, \ldots, y_n$ are independent.

Proof. Suppose $\{y_1, \ldots, y_n\}$ dependent; this would imply the existence of $z_1, \ldots, z_{n-1}$ such that $\{y_1, \ldots, y_n\} \subseteq V\{z_1, \ldots, z_{n-1}\}$; but, as $y_1, \ldots, y_{n-1}$ are independent and contained in $V\{z_1, \ldots, z_{n-1}\}$, we have, by Theorem 5.1, $V\{y_1, \ldots, y_{n-1}\} = V\{z_1, \ldots, z_{n-1}\}$; as $y_n \in V\{z_1, \ldots, z_{n-1}\}$ a contradiction arises.□

> *Theorem 5.2.* If $V \subseteq M$ is an n-dimensional manifold (with $n \geqslant 1$) and if $\{x_1, \ldots, x_r\}$ is an r-tuple of independent points of V, with $r \leqslant n$, then there exists an x_{r+1} in V such that $\{x_1, \ldots, x_r, x_{r+1}\}$ is an $(r+1)$-tuple of independent points.

Proof. $V\{x_1, \ldots, x_r\}$ cannot coincide with V; for, since V has dimension n, there must exist an $(n+1)$-tuple of independent points $\{z_1, \ldots, z_{n+1}\}$ such that $V\{z_1, \ldots, z_{n+1}\} = V$. $V\{x_1, \ldots, x_r\} = V$, with $r \leqslant n$, would contradict the independence property of the z_i; then pick an $x_{r+1} \notin V\{x_1, \ldots, x_r\}$ and belonging to V; by the corollary just proved, $x_1, \ldots, x_r, x_{r+1}$ are independent.□

It is obvious that, in the case of a manifold with infinite dimension, a completely analogous theorem warrants the possibility of finding in it an arbitrarily high number of independent points.

> *Lemma 5.6.* Let A be a subset of M, such that VA is of finite dimension n and A is open in the topology $\mathscr{T}_1(VA)$ induced on VA by $\mathscr{T}_1 \equiv \mathscr{T}_I$. If $\{x_1, \ldots, x_r\}$ is an r-tuple of independent points in A, with $r \leqslant n$, then there exists in A a point x_{r+1} such that $x_1, \ldots, x_r, x_{r+1}$ are independent.

Proof. Use Theorem 5.2 and find a point $x'_{r+1} \in VA$ such that $x_1, \ldots, x_r, x'_{r+1}$ are independent points; consider the segment $x_1x'_{r+1}$ and find on it a point $x_{r+1} \in A$ and such that $x_1x_{r+1} \subseteq A$; x_{r+1} is independent from $x_1, \ldots, x_r$: for, otherwise x'_{r+1}, which lies on a prolongation of x_1x_{r+1}, would lie in $V\{x_1, \ldots, x_r\}$ and this is absurd; x_{r+1} is the point we were looking for.□

We now introduce the following definition:

> *Definition.* If $x_1, \ldots, x_{r+1}$ are $r+1$ independent points, we call $x_1 \cdots x_{r+1}$ the r simplex with vertices $x_1, \ldots, x_{r+1}$ and $[x_1 \cdots x_{r+1}]$ the extended r simplex with the same vertices.

We can now prove the following lemma:

> *Lemma 5.7.* If V is an n-dimensional manifold of M, then any n simplex $x_1 \cdots x_{n+1}$ contained in V is open for the topology $\mathscr{T}_1(V)$. As $x_1 \cdots x_{n+1}$ is convex, it is an open set also for $\mathscr{T}_K(V)$.

Proof. This is done by induction in the following sense: Lemma 4.8 gives us the result for a two-dimensional manifold; but it also means, via Lemma 5.1, that, if we have an n-dimensional manifold and a nontrivial triangle in it, xyz, then it is open in the topology induced by $\mathscr{T}_K$ on $V\{x, y, z\}$. So, let us now assume that, in V, every $(r-1)$ simplex $x_1 \cdots x_r$, for a fixed $r(\leqslant n)$ is open in $V\{x_1, \ldots, x_r\}$; we will prove that then every r simplex $y_1 \cdots y_{r+1}$ is open in $V\{y_1, \ldots, y_{r+1}\}$. This will yield in particular the result to be proved. Consider (see Figure 12) $y_1 \cdots y_{r+1} = \{y_1\}(y_2 \cdots y_{r+1})$, a point $x \in y_1 \cdots y_{r+1}$ and any point $z \in V\{y_1, \ldots, y_{r+1}\}$; then, by virtue of Lemma 5.1 two points $u, v \in V\{y_1, \ldots, y_{r+1}\}$ exist such that $v \in uz$; as x and u belong to $\{y_1\}(y_2 \cdots y_{r+1})$, two points $x', u' \in y_2 \cdots y_{r+1}$ exist such that $x \in y_1x', u \in y_1u'$; now the induction hypothesis and Axiom 6 imply that a point $w' \in y_2 \cdots y_{r+1}$ exists such that $x' \in u'w'$. Consider $y_1u'w'$; by the associativity property, we know there exists $w \in y_1w'$ such that $x \in uw$; now Theorem 4.4, applied to zuw, leads to the existence of $z' \in xz \cap y_1 \cdots y_{r+1}$.□

The last-proved lemma aims obviously at reaching the equivalence proof between $\mathscr{T}_K(V)$ and the topology generated, in an n-dimensional manifold V,

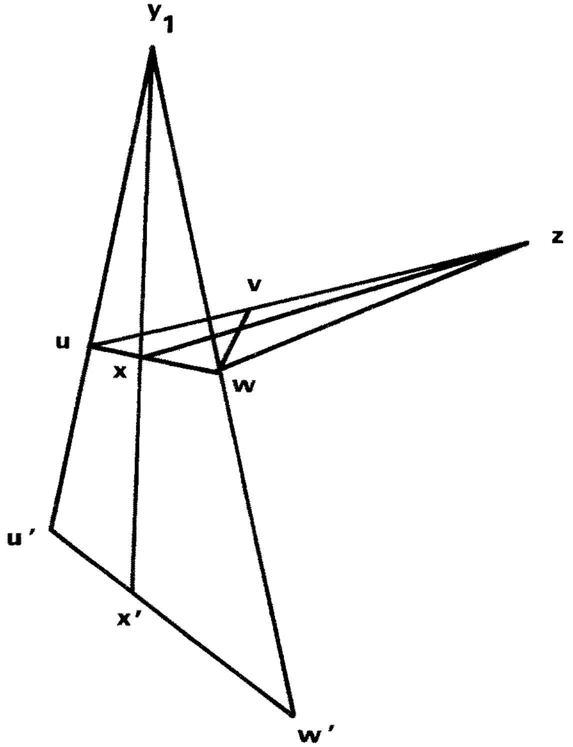

Figure 12.

by the family $\mathscr{B}_{\Sigma_n}(V)$ of all its n simplexes; we only need a further lemma, which is, in a sense, a strengthening of Lemma 5.6.

Lemma 5.8. Let V be an n-dimensional manifold of M; let B be an element of $\mathscr{B}_K$, basis for $\mathscr{T}_K(V)$, and let $x \in B$; then there exists a simplex Σ of $\mathscr{B}_{\Sigma_n}(V)$ such that $x \in \Sigma \subset B$.

Proof. Apply Lemma 5.6 to $x \in B$ n times and find $xy_1 \cdots y_n$, with $x, y_1, \ldots, y_n$ independent points of B; as B is convex $xy_1 \cdots y_n \subset B$; then take $z' \in y_1 \cdots y_n$ and, as B is open, $z \in B$ such that $x \in zz'$; obviously $z \notin y_1 \cdots y_n$ so that $z, y_1, \ldots, y_n$ are independent; besides, owing to the convexity of simplexes and of B, we have $x \in zy_1 \cdots y_n \subseteq B$.□

It remains to be checked that the family $\mathscr{B}_{\Sigma_n}$ is indeed a basis for a topology: This is obvious since (i) if Σ_1, Σ_2 belong to $\mathscr{B}_{\Sigma_n}$, then $\Sigma_1 \cap \Sigma_2$ is open and convex (for $\mathscr{T}_K$), and then, by Lemma 5.8, for any $x \in \Sigma_1 \cap \Sigma_2$ there is Σ_3 such that $x \in \Sigma_3 \subseteq \Sigma_1 \cap \Sigma_2$; (ii) for any $x \in M$ there is a $\Sigma \in \mathscr{B}_{\Sigma_n}$ that contains it (the proof is completely analogous to that of Lemma 5.8).

The result can now be stated in the following theorem:

Theorem 5.3. The topology generated on an n-dimensional manifold V by $\mathscr{B}_{\Sigma_n}(V)$ is $\mathscr{T}_K(V)$, that is, $\mathscr{B}_{\Sigma_n}(V)$ is a basis for the topology $\mathscr{T}_K(V)$.

One remarkable feature of the structure here exhibited consists in the possibility of obtaining the main results concerning the theory of affine spaces on the basis of relatively few axioms, concerning the order structure of segments and properties of their prolongations.

The analysis can be carried farther, as is easily understood, but we think it is already evident what the relevant implications concerning the space-time structure may be. Very roughly speaking, we have shown that the very simple-minded assumption about the possibility (easily implementable by means of elementary physical experiments, at least *locally*) of assigning a "segment" to each couple of events is quite far-reaching for the geometric structure of space-time.

Acknowledgement

A useful discussion with Prof. G. Meloni is gratefully acknowledged.

Reference

Birkhoff, G. (1973). *Lattice theory.* (American Mathematical Society Colloquium Publications, Providence, R.I.), 3rd ed., Chap. VIII, Theorem 23.